# Capacitance Computations in a Multilayered Dielectric Medium Using Closed-Form Spatial Green's Functions

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*Abstract*—**An efficient method to compute the 2-D and 3-D capacitance matrix of multiconductor interconnects in a multilayered dielectric medium is presented. The method is applicable to conductors of arbitrary polygon shape embedded in a multilayered dielectric medium with possible ground planes on the top or bottom of the dielectric layers. The computation time required to evaluate the space-domain Green's function for the multilayered medium, which involves an infinite summation, has been greatly reduced by obtaining a closed form, which is derived by approximating the Green's function using a finite number of images in the spectral domain. The corresponding space-domain Green's functions are then obtained using the proper closed-form integrations. The elements of the moment matrix are computed using the closed-form formulation, avoiding any numerical integration. The presented method showed good agreement when compared with other published results.**

## I. Introduction

The computation of parasitic capacitance coefficients in multilayered media is most commonly performed using an integral equation [1]-[4]. One major limitation of this approach is the infinite summation involved in the computation of the Green's function and the inexistence of a closed-form expression in the space domain. As noted in [2], for N layers, the expression for the Green's function would consist of N-1 infinite series. Alternatively, the free-space Green's function is used in [2] to avoid infinite series, but additional unknown charges on the dielectric interface and ground planes, on top of the unknown charges on the conductor surface, must be included.

Yet another approach to avoid an infinite summation is to solve the integral equation in the spectral domain (SDA), where the Green's function is in a closed form; however, this approach can not be applied to general problems, e. g., conductor with a finite thickness. In this paper, the Green's function for the layered medium is approximated in the spectral domain using the exponential functions, which is equivalent to a finite number of weighted real images in the space domain. Although the complex-valued exponentials, which are often used in a nonquasi-TEM analysis [5], can also be employed to reduce the number of weighted images, the real-valued exponentials are sufficient to approximate for quasi-TEM applications, and it further avoids the use of expensive complex operations. Since the spectral-domain representations of the Green's function for 2-D and 3-D cases are identical, the approximation is only performed once for both cases. After determining the equivalent weighted images in

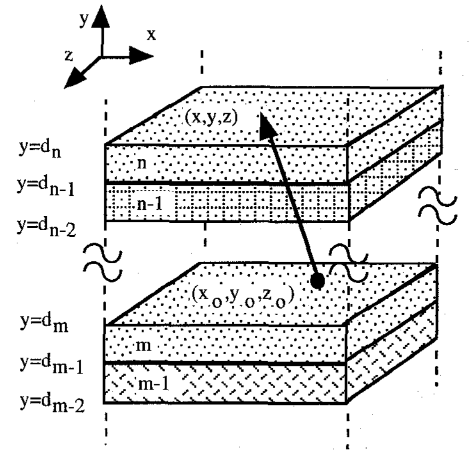



Fig. 1. The geometric configuration used for determining the Green's function.

cases. After determining the equivalent weighted images in
the spectral domain, they can be directly used to evaluate the Green's function in the space domain.

## II. Approximation of the Green's Function in the Spectral Domain

Consider a unit point charge located at the mth layer at $(x_o,y_o,z_o)$ (Fig. 1). Noting that the dielectric medium is uniform in two directions, we can represent the Green's function and the point source in the spectral domain in terms of its transforms in the x and $z$ directions. Then, solving the corresponding spectral-domain Laplace's equation with appropriate boundary conditions at the dielectric interfaces, the spectral-domain Green's function can be found as

$$\tilde{G}(\gamma, y|r_o) = \frac{1}{2\varepsilon_m\gamma}\Big(K^+_{m,n,1}e^{\gamma(y+y_o-2d_n)} + K^+_{m,n,2}e^{\gamma(+y-y_o+2(d_{m-1}-d_n))} + K^+_{m,n,3}e^{\gamma(-y+y_o)} + K^+_{m,n,4}e^{\gamma(-y-y_o+2d_{m-1})}\Big) \quad y \geq y_o \tag{1}$$

where

$$K^{+}_{m,n,1} = M_m \tilde{\Gamma}_{n,n+1} \prod_{j=m}^{n-1} S^{+}_{j,j+1} \quad (2a)$$

$$K^{+}_{m,n,2} = M_m \tilde{\Gamma}_{n,n+1} \tilde{\Gamma}_{m,m-1} \prod_{j=m}^{n-1} S^{+}_{j,j+1} \quad (2b)$$

$$K^{+}_{m,n,3} = M_m \prod_{j=m}^{n-1} S^{+}_{j,j+1} \quad (2c)$$

$$K^{+}_{m,n,4} = M_m \tilde{\Gamma}_{m,m-1} \prod_{j=m}^{n-1} S^{+}_{j,j+1} \qquad y \geq y_o \quad (2d)$$

$$S^{+}_{j,j+1} = \frac{T_{j,j+1}}{1 - \Gamma_{j+1,j} \tilde{\Gamma}_{j+1,j+2} e^{2\gamma(d_j - d_{j+1})}} \quad (3)$$

$$\tilde{\Gamma}_{j,j+1} = \frac{\Gamma_{j,j+1} + \tilde{\Gamma}_{j+1,j+2} e^{2\gamma(d_j - d_{j+1})}}{1 + \Gamma_{j,j+1} \tilde{\Gamma}_{j+1,j+2} e^{2\gamma(d_j - d_{j+1})}} \quad (4)$$

$$\Gamma_{i,j} = \frac{\varepsilon_i - \varepsilon_j}{\varepsilon_i + \varepsilon_j} \qquad T_{i,j} = \frac{2\varepsilon_i}{\varepsilon_i + \varepsilon_j} \quad (5)$$

$$\gamma = \sqrt{\alpha^2 + \beta^2} \quad (6)$$

where $\tilde{G}(\alpha, \beta, y|r_o)$ is the spectral domain Green's function and $\alpha$ and $\beta$ are the transform constants associated with the $x$ and $z$ directions, respectively. The subscript $m$ denotes the layer where the source is located (source point) while the subscript $n$ denotes the layer where the Green's function is evaluated (observation point). $\tilde{\Gamma}_{j,j+1}$ is the generalized reflection coefficient, which is the ratio of the amplitudes of voltages at $y=dn$ due to the image charges located above and below the jth layer. $\tilde{\Gamma}_{j,j+1}$ takes the value of 0 or -1 if the jth layer is a half space, or the (j+1)th layer is a ground plane, respectively. The superscript + is used to denote the quantities related with the Green's function for $y \geq y_o$. The similar expressions can be obtained for $y \leq y_o$.

In Equation (1), $K^{+}_{m,n,i}$ is not a function of $y$ and $y_o$, and the determination of the closed-form space-domain Green's function can now be preceded by approximating the coefficient functions $K^{+}_{m,n,i}$ of the exponential terms. It is important to notice that we have also factored out other exponential terms which dominate the behavior of the function for large $\gamma$, this, in turn, will assure an accurate approximation of the function at the short range of distance in the space domain.

A physically intuitive approach to approximate the potential due to a charge in the layered medium is the use of a finite number of the weighted image charges in the homogenous medium, which is equivalent to approximating the coefficient functions $K^{+}_{m,n,i}$ with exponential functions in the spectral domain. The equivalence between the weighted image charges and exponential functions will be shown later in Equations (15) and (16). Noting that the coefficient func

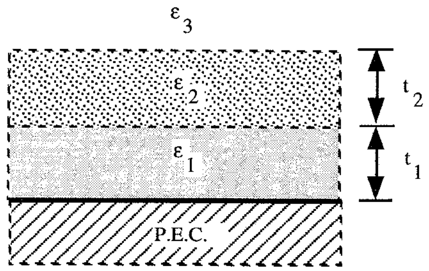



Fig. 2. The geometry used to test the approximated Green's function.

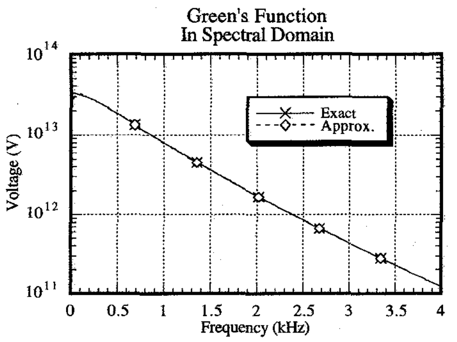
tions $K^{+}_{m,n,i}$ are real-valued and nonoscillatory, the real-valued exponentials are sufficient to approximate the coefficient functions and avoid any complex operations. A relaxation algorithm based on curve fitting in [6] is used here. Although this method is simple and iterative in nature, it converges to reasonable accuracy in a few iterations, and requires less computation time as compared to the other existing methods, such as, the generalized pencil-of-function (GPOF) and the Prony methods.

In the process of the approximation, the symmetry of the coefficient functions $K^{+}_{m,n,i}$ is explored to increase the accuracy of the approximated functions and to reduce the computation time. The same approximation procedure can be applied for the case for $y \leq y_o$.

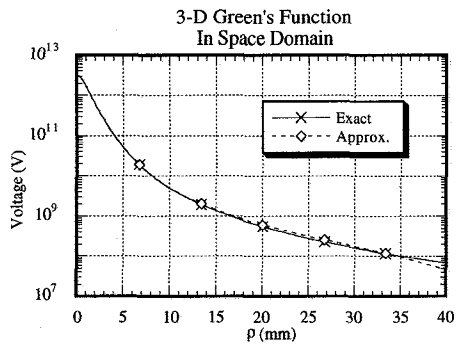
The Green's function for the structure shown in Fig. 2 is approximated and compared with the exact Green's function in the spectral domain. The dielectric constants $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are taken to be $2\varepsilon_o$, $5\varepsilon_o$, and $\varepsilon_o$, accordingly. The thicknesses of the layers $t_1$ and $t_2$ are 0.6 mm and 1.0 mm, and the source and observation points, $y$ and $y_o$, are 0.6 mm and 1.6mm, respectively. A maximum frequency of 10 kHz and an average number of 7 exponentials were used to approximate the Green's function. As shown in Fig. 3(a), the approximated results agree with the original one, except for very small arguments where the approximation errors are amplified by the singularity factor (see Equation (1)). It is important to observe that although the exponential approximation might fail to approximate for the large argument due to its fast decaying nature, by extracting the asymptotic value and the exponential factor from the coefficient function, the limiting behavior of the overall approximated Green's function would still remain accurate. Hence, it is expected that the approximated Green's function will be accurate for the short distance range in the space domain.

After approximating the Green's function in the spectral domain, one can obtain the space domain Green's function using the following identity:

$$\frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\alpha d\beta e^{-j(\alpha x + \beta z)} \frac{e^{-\gamma|y|}}{\sqrt{\alpha^2 + \beta^2}} \quad (7)$$



(a)



(b)

Fig. 3. Comparison of approximated and exact Green's functions in (a) the spectral domain and (b) the space domain for the 3-D case.

which can be viewed as the static version of Weyl's identity and can be derived by considering the potential due to the unit point charge. Thus, in the space domain, the approximated Green's function can be written as

$$G(r|r_o) = \frac{1}{4\pi\varepsilon_m} \sum_{j=1}^{4} f_j^{\pm}(r|r_o) \tag{8}$$

where the superscript + for $y \geq y_o$ and - for $y \leq y_o$. For $j$=1, the expression for $f_j^{+}(r|r_o)$ is given by

$$f_1^{+}(r|r_o) = K_{m,n,1}^{+,\infty} \frac{1}{\sqrt{(x-x_o)^2 + (y-y_o-2d_n)^2 + (z+z_o)^2}} + \sum_{i=1}^{N_{m,n,1}^{+}} C_{m,n,1}^{+,i} \frac{1}{\sqrt{(x-x_o)^2 + (y-y_o-2d_n+a_{m,n,1}^{+,i})^2 + (z+z_o)^2}} \tag{9}$$

Here, $K_{m,n,j}^{+,\infty}$ denotes the asymptotic value of $K_{m,n,j}^{+}$ and we have assumed that $K_{m,n,j}^{+}$ is approximated by

$$K_{m,n,j}^{+}(\gamma) = \sum_{i=1}^{N_{m,n,j}^{+}} C_{m,n,j}^{+,i} e^{a_{m,n,j}^{+,i}\gamma} \tag{10}$$

where $N_{m,n,j}^{+}$ is the number of exponential functions used to approximate the coefficient function $K_{m,n,j}^{+}$.

The identical expression can be derived for the two-dimensional Green's function where the transform variable $\gamma$ will be associated with $x$, and the following identity can be used to obtain the space-domain expression:

$$-\ln(\rho) = -\ln\left(\sqrt{x^2+y^2}\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\gamma e^{-j\gamma x} \frac{e^{-\gamma|y|}}{\gamma} \tag{11}$$

Thus, the space domain expression similar to Equation (8) can be obtained as follows:

$$G(\rho|\rho_o) = \frac{1}{2\pi\varepsilon_m} \sum_{j=1}^{4} f_j^{\pm}(\rho|\rho_o) \tag{12}$$

where, $f_j^{+}(r|r_o)$ is the corresponding logarithmic function.

The approximated Green's function is also compared in the space domain. The same structure is used as in the previous one with $\varepsilon_2=\varepsilon_1=2\varepsilon_0$, and z and $z_o$ were 0.6 mm and 1.6 mm, respectively. A maximum frequency of 4 kHz and 5 exponentials were used in the approximation. The exact Green's function is obtained by applying the image principle in the space domain. Again, both Green's functions are plotted in Fig. 3(b) and showed a good agreement.

Although it is not clearly shown in Fig. 3(b) the relative approximation error was found to increase monotonically as the distance between the source and observation points increased; however, the error still remained small for a practical range. In this particular case, the relative error at $\rho$=100 mm was less than 1 percent.

## IV. Solution Method for the Integral Equation

The integral equation relating the electrostatic potential $V(r)$ to the charge density $\rho(r)$ is

$$V(r) = \int_{\Omega} G(r,r')\rho(r')dr' \tag{13}$$

where $G(r,r')$ is the Green's function for the multilayered medium, and $\Omega$ is surfaces or cross-sectional boundaries of all conductors for 2-D or 3-D problems, respectively. Employing the pulse-type basis functions and the point matching technique, Equation (13) becomes

$$V_i = \sum_{k=1}^{N_T} q_k \int_{\Omega_k} G(r_i|r)d\Omega, \quad i = 1, 2, \cdots, N_T \tag{13}$$

where $q_k$ is the unknown coefficient to be determined, and $N_T$ is the total number of basis functions used. The closed-form formulae for the evaluation of Equation (13) over an arbitrary polygon patch and a line segment are given in [7]. Thus, the resulting matrix form Equation (13) can be constructed without any numerical integration.

## III. NUMERICAL EXAMPLES

The computer program was developed based on the discussed method, and it is capable of handling an arbitrary number of dielectric layers and conductors and designed to read mesh data from a conventional mesh generator to allow computation of the complex geometries and meshes. First, the capacitance matrix for three conductors with finite thicknesses in a layered medium, shown in Fig. 4, is computed. The number of basis functions used was 100 for each conductor, and the maximum number of exponentials used to approximate the coefficient function $K^{\pm}_{m,n,i}$ was 5. Comparison with results in [3] is shown in Table I. In [3], the spectral-domain Green's function is numerically integrated to convert to the space domain using a Gaussian quadrature formula in conjunction with analytical asymptotic extraction. For the next numerical example, the equivalent capacitances for a microstrip crossover is considered. The same geometry used in [4] was considered, where $\varepsilon_1$ and $\varepsilon_2$ were 2 and 1, and the heights of the lower and upper microstrip lines were 4 mm and 6 mm, respectively. The widths of both strips were 0.16 mm. Table II shows the comparison with [4]. In Table II, $c_1'$ and $c_2'$ are the capacitances per unit length of the isolated lines with radii equal to 0.25W and $\varepsilon_1=\varepsilon_2=1$. The Green's function used in [4] is based on the image principle in the space domain and involves an infinite summation. According to [4], only ten terms were sufficient to evaluate the infinite summation for this particular structure, where lines are extremely narrow. However, it can be easily seen from the expression of their Green's function the number of terms required will be much larger for wider microstrip lines.

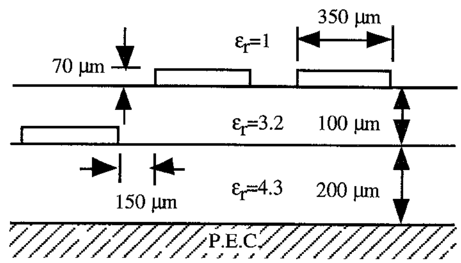



fig 4. Three conductors in a layered medium All dimensions of conductors and spacings are identical

Table I Comparison of the capacitance matrix for the three conductor structure

| | | | | |
|---|---|---|---|---|
| Delabare et al. [3] | 142.09 | −21.765 | −0.8920 | (pF/m) |
| | −21.733 | 93.529 | −18.098 | |
| | −0.8900 | −18.097 | 87.962 | |
| Computataion | 145.33 | −23.630 | −1.4124 | (pF/m) |
| | −22.512 | 93.774 | −17.870 | |
| | −1.3244 | −17.876 | 87.876 | |

Table II. Comparison results for a microstrip crossover

| | $c_1^s / h_1 c_1'$ | $c_2^s / h_1 c_2'$ | $c^m / h_1 * c_2'$ |
|---|---|---|---|
| Papatheodorou et al. [2] | -1.345 | -1.296 | 1.672 |
| Computation | -1.202 | -1 288 | 1 652 |

## CONCLUSIONS AND FUTURE WORK

In the capacitance computation, the time to construct the moment matrix takes the major portion of the computation time, the presented method significantly reduced this computation time by obtaining the closed-form Green's function numerically. The method was verified with numerous other published results, and only two rather complex examples are presented here. Finally, it is emphasized again that the presented method avoids any numerical integration or infinite summations.

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